A NOTE ON COHOMOTOPY SETS

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1. Introduction

For a space X, we will call a rational cohomology class $\alpha(\in H^n(X: Q))$ to be representable if there exists a map $f: X \longrightarrow S^n$ such that α is contained in $f^*(H^n(S^n: Q))$. Our purpose is to give a characterization of representable classes. H. Oshima defined in [1] the concept of n-th Co-degree $D_n(X)$, namely

$$D_n(X) = \{f^* | f: X \longrightarrow S^n\} \subset Hom(\pi_n(X), \pi_n(S^n)).$$

We denote by $D_n(X)_Q$ a rationalization of $D_n(X)$, i.e.

$$D_n(X)_Q = \{f^* | f: X \longrightarrow S^n\} \subset Hom(\pi_n(X) \otimes Q, \pi_n(S^n) \otimes Q).$$

Then, via Hurewicz homomorphism, we have a homomorphism

$$\begin{aligned} H^n(X: Q) = Hom(H_n(X: Z), Q) &\longrightarrow \\ Hom(\pi_n(X) \otimes Q, Q) &= Hom(\pi_n(X) \otimes Q, \pi_n(S^n) \otimes Q) \end{aligned}$$

which sends {representable classes} to $D_n(X)_Q$.

For example the above fact contains that the triviality of $D_n(X)$ is equivalent to {representable classes} $= \{0\}$.

In this note we prove

Theorem *For a finite CW-complex X, $\alpha(\in H^n(X: Q))$ is representable if and only if $\alpha^2 = 0$.*

Corollary 1. *$D_n(X)$ is non trivial if and only if there exists a class $\alpha \neq 0$ with $\alpha^2 = 0$.*

This corollary can be restated as follows:

Corollary 2. *$D_n(X) = \{0\}$ if and only if $\alpha^2 \neq 0$ for all non zero n-dim rational cohomology classes α of X.*

2. Lemmas

Let X be a connected finite CW-complex which has a decomposition:

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$$X=\{\vee S^n\}\vee\{\vee M(n, Z_m)\}\cup\{e^{n+1}\}\cup\cdots\{e^{2n-1}\}\cup\{e_k^{2n}\}$$

where $M(n, Z_m)$ denotes the Moore space of type (n, Z_m).

First we note the following trivial

Lemma 1. *If $H^n(X: Q)=\{0\}$ then $D_n(X)_Q=\{0\}$.*

Thus, since $M(n, Z_m)$ is rationally trivial we may assume that X dose not contain the part $\{\vee M(n, Z_m)\}$.

Let f: $\vee S_j^n \longrightarrow S^n$ be a map of degree d_j on each S_j^n. *Since* $\pi_{n+k}(S^n)$ is finite for $0<k<n-1$ and also $k=n-1$ and also $k=n-1$ if n is odd we have

Lemma 2. *There exists an integer $d\neq 0$ such that the composite:*

$$\vee S_j^n \xrightarrow{f} S^n \xrightarrow{d} S^n$$

is extendable over X_{2n-1} (the $(2n-1)$-skelton of X) and over $X_{2n}=X$ for odd n.

In the following we suppose that n is even. Let $\tilde{f}$ be an extension as above and let $\alpha_k(\in\pi_{2n-1}(X_{2n-1}))$ be the attaching class for the k-th $2n$-cell e_k^{2n}. Then we may assume that $\tilde{f}_*(\alpha_k)=x_k[\iota_n, \iota_n]$ for some integer x_k and the generator ι_n of $\pi_n(S^n)$.

Lemma 3. $\Sigma\, x_k e_k^{2n}=\tilde{f}^*(S^n)^2=d^2\tilde{f}^*(S^n)$ *in* $H^*(X: Z)$.

Proof. Let $S^n\cup\{e_k^{2n}\}$ be the complex which is obtained from attaching each $2n$-cell with $[\iota_n, \iota_n]$ and let h be a map: $X\longrightarrow S^n\cup\{e_k^{2n}\}$ such that

(1) $h|X_{2n-1}=\tilde{f}$

(2) h is of degree x_k on e_k^{2n}.

Since $\tilde{f}^*(\alpha_k)=x_k[\iota_n, \iota_n]$ means that

$$h^*(S^n)^2=h^*((S^n)^2)=h^*(\Sigma\, e_k^{2n})=\Sigma\, x_k e_k^{2n}$$

the proof follows from $h^*(S^n)^2=\tilde{f}^*(S^n)^2$.

Lemma 4. *If e_k^{2n} represents a torsion element of $H^{2n}(X: Z)$ then the restriction $(N\iota_n\tilde{f})|X_{2n-2}$ is extendable over $X_{2n-1}\cup\{e_k^{2n}\}$ for some integer $N\neq 0$.*

Proof. It may be regarded that $\partial e_k^{2n}=se_{2n-1}$ for a cell e_{2n-1}. Let γ be a map $X_{2n-1}\longrightarrow X_{2n-1}\vee S^{2n-1}$ which is obtained from identifying the equator of e^{2n-1} to a point and consider the composite:

$$X_{2n-1}\xrightarrow{\gamma} X_{2n-1}\vee S^{2n-1}\xrightarrow{\tilde{f}\vee\beta} S^n$$

for any map β: $S^{2n-1}\longrightarrow S^n$.

Then we have

$$(\tilde{f}\vee\beta)_*(\alpha_k)=(\tilde{f}\vee\beta)_*(\alpha_k+s\iota_{2n-1})=\tilde{f}_*(\alpha_k)+s\beta=x_k[\iota_n, \iota_n]+s\beta$$

Hence if we take a multiple of s as N it holds

$$(N\iota_n)_*(\tilde{f}\vee\beta)_*(\alpha_k)=0 \quad (\beta=-(n/s)Nx_k[\iota_n, \iota_n]).$$

This shows the proof.

Lemma 5. *Let X be a $(n-1)$-connected finite CW-complex of dimension $2n$ whose cohomology is torsion free in dimension $2n$.*

Then, for a map g: $X_{2n-1} \longrightarrow S^n$, there exists an integer $d \neq 0$ such that the composite $(d\iota_n)g$ is extendable over X if and only if $g^*(S^n)^2=0$ in $H^*(X: Q)$.

Proof. First suppose $g^*(S^n)^2=0$. By lemma 3 there exists an integer $d \neq 0$, and $g_d=(d\iota_n)g$ has the cohomology class $g_d^*(S^n)^2$ as an obstruction for extendability. Since $g_d^*(S^n)^2=d^2g^*(S^n)^2=0$, g_d is extendable over X.

Conversely, suppose that g_d is extendable over X for some integer $d \neq 0$.

Again, by lemma 3, there exists an integer $m \neq 0$ such that $(m\iota_n)g_d$ has $((m\iota_n)g_d)^*(S^n)^2$ as an obstruction.

Since g_d is extendable over X, $(m\iota_n)g_d$ is also extendable over X, i.e.

$$0=((m\iota_n)g_d)^*(S^n)^2=m^2d^2g^*(S^n)^2.$$

Clearly this shows $g^*(S^n)^2=0$.

3. Proof of Theorem

Let X be a connected finite CW-complex.

Lemma 6. *For a map h: $X_{2n} \longrightarrow S^n$ there exists an integer $N \neq 0$ such that $(N\iota_n)h$ is extendable over X.*

Proof. This lemma immediately follows from the finiteness of homotopy groups $\pi_{2n+k}(S^n)$ $(0 \leq k)$ and the formula (8.9) in [2].

Specially we have

Lemma 7. *Let i: $X_{2n} \longrightarrow X$ be the inclusion map.*

Since i^*: $H^n(X: Q) \longrightarrow H^n(X_{2n}: Q)$ is an isomorphism, it holds that $\alpha(\in H^n(X: Q))$ is representable if and only if $i^*(\alpha)$ is representable.

Now we prove the fundamental lemma.

Lemma 8. *Let X be a $(n-1)$-connected finite CW-complex of dimension $2n$. A cohomology class $\alpha(\in H^n(X: Q))$ is representable if and only if $\alpha^2=0$.*

Proof.

(1) $n \equiv 1 \mod 2$

In this case it always holds $\alpha^2=0$, and on the other hand any element of $H^n(X: Q)$ is representable by lemma 2.

(2) $n \equiv 0 \mod 2$

In this case the proof follows from lemma 4 and 5.

At last we give the proof of Theorem. Let α be a n-dim rational cohomology class of X and let i be the inclusion: $X_{2n} \longrightarrow X$.

By lemma 7 α is representable if and only if $i^*(\alpha)$ is representable. And moreover, by considering the diagram:

$$\begin{array}{ccc} X_{2n} & \longrightarrow & S^n \\ \downarrow{\scriptstyle p} & & \\ X_{2n}/X_{n-1} & & \end{array}$$

we can know that $i^*(\alpha)$ is representable if and only if it is the p^*-image of a representable class β of $H^n(X_{2n}/X_{n-1})$.

By lemma 8 $\beta(\in H^n(X_{2n}/X_{n-1}: Q))$ is representable if and only only if $\beta^2=0$.

Then from the equality: $p^*(\beta^2)=p^*(\beta)^2=i^*(\alpha)^2$ and the injectivity of the homomorphism $p^*: H^{2n}(X_{2n}/X_{n-1}: Q) \longrightarrow H^{2n}(X_{2n}: Q)$ we obtain that $\beta^2=0$ is equivalent to $i^*(\alpha)^2=0$.

Thus α is representable if and only if $i^*(\alpha)^2=0$.

Since $i^*: H^{2n}(X: Q) \longrightarrow H^{2n}(X_{2n}: Q)$ is injective the proof is completed.

References

[1] H. Oshima and K. Takahara, *Cohomotopy of Lie groups*, Osaka J. Math. vol. 28, pp. 213–221 (1991).

[2] G. W. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, GTM. 61 (1978).